

# The class of distributions associated with the generalized Pollaczek-Khinchine formula

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## Abstract

The goal is to identify the class of distributions to which the distribution of the maximum of a Lévy process with no negative jumps and negative mean (equivalently, the stationary distribution of the reflected process) belongs. An explicit new distributional identity is obtained for the case where the Lévy process is an independent sum of a Brownian motion and a general subordinator (nondecreasing Lévy process) in terms of a geometrically distributed sum of independent random variables. This generalizes both the distributional form of the standard Pollaczek-Khinchine formula for stationary workload distribution in the M/G/1 queue and the exponential stationary distribution of a reflected Brownian motion.

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## 1 Introduction and preliminaries

Let  $X = \{X_t \mid t \geq 0\}$  be a Lévy process with no negative jumps. It is standard knowledge that in this case  $Ee^{-\alpha X_t} = e^{\varphi(\alpha)t}$  is finite for all  $\alpha \geq 0$  and that

$$\varphi(\alpha) = b\alpha + \frac{\sigma^2}{2}\alpha^2 + \int_{(0,\infty)} (e^{-\alpha x} - 1 + \alpha x 1_{\{x \leq 1\}}) \nu(dx) \quad (1)$$

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where  $b$  is real,  $\sigma$  is nonnegative and  $\nu$  is a measure satisfying

$$\int_{(0,\infty)} (x^2 \wedge 1) \nu(dx) < \infty \quad (2)$$

where  $a \wedge b = \min(a, b)$ .

It is also well known that  $-EX_t/t = \varphi'(0) = -b + \int_{(1,\infty)} x\nu(dx)$  and in particular  $EX_t$  is well defined and can be either finite or  $+\infty$  but never  $-\infty$ . In particular if  $EX_t < 0$  (equivalently  $\varphi'(0) > 0$ ) then  $\int_{(1,\infty)} x\nu(dx) < b$  so that in particular  $b > 0$  and  $\int_{(1,\infty)} x\nu(dx) < \infty$ . Therefore, in this case  $\varphi$  has the following form

$$\varphi(\alpha) = \mu\alpha + \frac{\sigma^2}{2}\alpha^2 + \int_{(0,\infty)} (e^{-\alpha x} - 1 + \alpha x) \nu(dx) \quad (3)$$

where  $\mu = \varphi'(0) = b - \int_{(1,\infty)} x\nu(dx) > 0$ .

When in addition  $\int_{(0,1]} x\nu(dx) < \infty$  then the Lévy process is an independent sum of a Brownian motion and a pure jump subordinator (nondecreasing Lévy process). In this case,  $\varphi$  becomes

$$\varphi(\alpha) = c\alpha + \frac{\sigma^2}{2}\alpha^2 - \int_{(0,\infty)} (1 - e^{-\alpha x}) \nu(dx) , \quad (4)$$

where  $c = \mu + \int_{(0,\infty)} x\nu(dx)$ .

It is also well known that for any Lévy process with no negative jumps and  $\varphi'(0) > 0$ , if we denote  $M = \sup_{t \geq 0} X_t$ , then

$$Ee^{-\alpha M} = \frac{\alpha\varphi'(0)}{\varphi(\alpha)} \quad (5)$$

and that the limiting and stationary distributions of the (Markov) process  $W_t = X_t + L_t$ , where  $L_t = -\inf_{0 \leq s \leq t} X_s$ , is the distribution of  $M$ . We (and others) call the formula  $\frac{\alpha\varphi'(0)}{\varphi(\alpha)}$  the *generalized Pollaczek-Khinchine (PK) formula*. The reason for the name is that for the special case of the M/G/1 queue, the underlying driving (Lévy) process is a compound Poisson process (a subordinator) minus  $t$  for which the Laplace-Stieltjes transform of the limiting and stationary distribution of the workload process is the celebrated Pollaczek-Khinchine formula and may be found in virtually all basic queueing theory textbooks.

There are various textbooks where Lévy processes are discussed and where the above results may be either found directly or concluded from (e.g., [1, 4, 8, 13] and pages 19-34 of [12]). There are quite a few different proofs in the literature for the generalized PK formula, mostly via the application of the Wiener-Hopf factorization (e.g., [6, 15]), weak convergence (e.g., [5, 14]) and martingales (e.g., [7]), but this is not the scope here.

For some recent work on the distribution (rather than Laplace-Stieltjes transform) of  $\alpha$ -stable Lévy processes see [2, 9, 10] and further references therein. For the case with phase type upward jumps (and general negative jumps) see [11]. For some other results see also [3]. These papers also include an extensive list of references to texts and further related literature.

One is not required to be a Lévy process expert to read this paper and all the knowledge which is needed for what follows is covered above. In particular, this may easily be taught in any course where Lévy processes are touched upon.

## 2 The case of an independent sum of a Brownian motion and a subordinator

Let  $X$  be a Lévy process which is an independent sum of a Brownian motion and a subordinator having a finite mean. In this case, as seen in the previous section, the (Laplace-Stieltjes) exponent can be written in the form

$$\varphi(\alpha) = c\alpha + \frac{\sigma^2\alpha^2}{2} - \int_{(0,\infty)} (1 - e^{-\alpha x})\nu(dx) , \quad (6)$$

where  $\bar{\nu} \equiv \int_{(0,\infty)} x\nu(dx) = \int_0^\infty \nu(x, \infty)dx < c$ . Denoting

$$F_e(\alpha) = \int_0^\infty e^{-\alpha x} \frac{\nu(x, \infty)}{\bar{\nu}} dx , \quad (7)$$

$\rho = \frac{\bar{\nu}}{c} < 1$  and  $\lambda = \frac{2c}{\sigma^2}$ , we observe that the exponent may be rewritten like this

$$\varphi(\alpha) = c\alpha \left( 1 + \frac{\alpha}{\lambda} - \rho F_e(\alpha) \right) \quad (8)$$

and in particular  $\varphi'(0) = c(1 - \rho)$  so that the generalized PK formula has the form

$$\frac{\alpha\varphi'(0)}{\varphi(\alpha)} = \frac{1 - \rho}{1 + \frac{\alpha}{\lambda} - \rho F_e(\alpha)} \quad (9)$$

If  $\sigma^2 = 0$  ( $\lambda = \infty$ ) and  $\nu(0, \infty) < \infty$  then one obtains the well known (original) PK formula for the M/G/1 queue. When  $\nu(0, \infty) = \infty$  it is interesting to observe that exactly the same formula is valid without change only that  $F_e$  is then the stationary excess life distribution associated with the jumps of the subordinator and unlike in the renewal process setting, its density approaches  $\infty$  in the neighborhood of zero. If the Lévy measure is zero then  $\rho = 0$  and the formula becomes the Laplace-Stieltjes transform (LST) of an exponential distribution with rate  $\lambda$ . This is also well known to be the LST of the stationary distribution associated with a one dimensional reflected Brownian motion

with negative drift. The interesting discovery is that these two results can be unified. To see this, note that

$$\begin{aligned} \frac{\alpha\varphi'(0)}{\varphi(\alpha)} &= \frac{\lambda}{\lambda + \alpha} \frac{1 - \rho}{1 - \rho F_e(\alpha) \frac{\lambda}{\lambda + \alpha}} \\ &= \sum_{n=0}^{\infty} (1 - \rho) \rho^n F_e^n(\alpha) \left( \frac{\lambda}{\lambda + \alpha} \right)^{n+1} \end{aligned} \quad (10)$$

and so we have the following result.

**Theorem 1** *For a Lévy process with no negative jumps satisfying  $\int_{(0,1]} x\nu(dx) < \infty$ ,  $\varphi'(0) > 0$  and with the notations defined above, let  $N \sim G(1 - \rho)$  in the sense that  $P[N = n] = (1 - \rho)\rho^n$ . Let  $X_0, X_1, X_2, \dots \sim \exp(\lambda)$  ( $= 0$  for  $\lambda = \infty$ ) and  $Y_1, Y_2, \dots \sim F_e$  and assume that  $N, X_0, X_1, \dots, Y_1, Y_2, \dots$  are all independent. Then  $\frac{\alpha\varphi'(0)}{\varphi(\alpha)}$  is the LST of the following random variable*

$$X_0 + \sum_{n=1}^N (X_n + Y_n) \quad (11)$$

where an empty sum is zero.

When  $\bar{\nu} \downarrow 0$  then we are left with  $X_0$  as expected (the Brownian motion case) and when  $\sigma^2 \downarrow 0$  we are left with  $\sum_{n=1}^N Y_n$  also as expected (distributional form of the PK formula).

An interesting special case occurs when the jumps of the Lévy process have a phase-type distribution. Then the residual life also has a phase-type distribution and thus also  $X_i + Y_i$  and  $X_0 + \sum_{n=1}^N (X_n + Y_n)$ . This is a special case of the results reported in [11] but with an easier proof but also a far less complicated setup.

It is very easy to check that also a converse holds.

**Theorem 2** *Assume that  $0 < p \leq 1$ ,  $0 < \lambda \leq \infty$ ,  $f$  is nonnegative, nonincreasing with  $\int_0^\infty f(y)dy = 1$  (possibly with  $f(x) \rightarrow \infty$  as  $x \downarrow 0$ ) and denote  $F(x) = \int_0^x f(y)dy$ . Let  $N \sim G(p)$ ,  $X_0, X_1, \dots \sim \exp(\lambda)$  ( $= 0$  for  $\lambda = \infty$ ) and  $Y_1, Y_2, \dots \sim F$  where all random variables are independent. Then there exists a Lévy process which is an independent sum of a Brownian motion and a subordinator having a negative mean for which  $\frac{\alpha\varphi'(0)}{\varphi(\alpha)}$  is the LST of*

$$X_0 + \sum_{n=1}^N (X_n + Y_n) , \quad (12)$$

where an empty sum is zero. This Lévy process is unique up to a constant time scale.

**Proof:** To see this we can simply perform reverse engineering. First assume without loss of generality that  $f$  is right continuous, otherwise we take its right continuous version which gives the same  $F$ . Let  $\nu((x, \infty)) = \beta f(x)$  for any constant  $\beta > 0$  and thus  $\nu((a, b]) = \beta(f(a) - f(b))$ , which uniquely characterizes  $\nu$ . Note that

$$\int_{(0, \infty)} x \nu(dx) = \int_0^\infty \nu((x, \infty)) dx = \int_0^\infty \beta f(x) dx = \beta < \infty \quad (13)$$

as required. Recalling that  $\rho = \frac{\bar{\nu}}{c}$ , that necessarily  $p = 1 - \rho$  and since  $\bar{\nu} = \beta$  we must set  $c = \frac{\beta}{1-p}$ . Now, if  $\lambda = \infty$  we set  $\sigma^2 = 0$  and otherwise, from  $\lambda = \frac{2c}{\sigma^2}$  we set  $\sigma^2 = \frac{2\beta}{\lambda(1-p)}$ . Finally,

$$\varphi'(0) = c - \bar{\nu} = \frac{\beta}{1-p} - \beta > 0 \quad (14)$$

so that all requirements are met.

The fact that this Lévy process is unique up to a constant time scale is evident since if

$$\frac{\alpha \varphi'(0)}{\varphi(\alpha)} = \frac{\alpha \psi'(0)}{\psi(\alpha)} \quad (15)$$

then necessarily  $\psi(\alpha) = \gamma \varphi(\alpha)$  for  $\gamma = \frac{\psi'(0)}{\varphi'(0)}$ , so that if  $X$  is a Lévy process with exponent  $\varphi$  and  $Y$  with  $\psi$ , then  $\{X_{\gamma t} | t \geq 0\}$  is distributed like  $Y$ . ■

### 3 The general spectrally positive case

The fact that  $\frac{\alpha \varphi'(0)}{\varphi(\alpha)}$  is the LST of some proper distribution for any Lévy process with no negative jumps and a negative mean is deduced indirectly from its various proofs which always exploit the connection with either the supremum or the reflected process. However, we are not aware of a more explicit and direct way of showing this than Theorems 1,2 together with the following.

Observe that if  $\int_{(0,1]} x \nu(dx) = \infty$  then the Lévy process is not a sum of a Brownian motion and a subordinator. In this case (since  $\varphi'(0) > 0$  and thus  $\int_{(1,\infty)} x \nu(dx) < \infty$ ), we recall from (3) in Section 1 that

$$\varphi(\alpha) = \mu\alpha + \frac{\sigma^2 \alpha^2}{2} + \int_{(0,\infty)} (e^{-\alpha x} - 1 + \alpha x) \nu(dx) \quad (16)$$

and that  $\varphi'(0) = \mu$ . Now, with

$$\varphi_\epsilon(\alpha) = \mu\alpha + \frac{\sigma^2 \alpha^2}{2} + \int_{(\epsilon,\infty)} (e^{-\alpha x} - 1 + \alpha x) \nu(dx) \quad (17)$$

for  $\epsilon > 0$ , we clearly have that  $\varphi_\epsilon(\alpha) \rightarrow \varphi(\alpha)$  as  $\epsilon \downarrow 0$ , that  $\varphi'_\epsilon(0) = \varphi'(0) = \mu$  and thus

$$\lim_{\epsilon \downarrow 0} \frac{\alpha \varphi'_\epsilon(0)}{\varphi_\epsilon(\alpha)} = \frac{\alpha \varphi'(0)}{\varphi(\alpha)}. \quad (18)$$

and that  $\frac{\alpha \varphi'_\epsilon(0)}{\varphi_\epsilon(\alpha)} \rightarrow 1$  as  $\alpha \downarrow 0$ .

Now  $\varphi_\epsilon$  is the exponent of a Lévy process which is an independent sum of a Brownian motion and a compound Poisson process for which we have identified the distribution associated with  $\frac{\alpha \varphi'_\epsilon(0)}{\varphi_\epsilon(\alpha)}$  in Theorem 1. From this the following is immediate.

**Theorem 3** *A distribution has the generalized PK LST for some Lévy process with no negative jumps and a negative mean if and only if it belongs to the closure of the family of distributions defined in Theorem 2.*

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